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The objective of this study was to determine ways of attaining a working competence in the arithmetic of the full real number system, and to explore means of removing some ineffective practice employed in instruction. A secondary objective was to determine the feasibility of developing infinite decimal arithmetic directly, without building on the arithmetic of fractions. The experimental testing of the classroom materials took place at the Middle School of the Laboratory Schools of the University of Chicago during 1964-65. Among the principal conclusions of this study are (1) working mastery of the real number system by appropriate use of its infinite decimal representation as a vehicle, and (2) this arithmetic can be built on or developed independently of fraction facts. (RP)

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INVESTIGATIONS ON THE FEASIBILITY OF SOME EXTENSIONS OF SCHOOL ARITHMETIC

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GEORGE KLEIN

May 1968

The research reported herein was performed pursuant to a contract with the Office of Education, U. S. Department of Health, Education, and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their professional judgment in the conduct of the project. Points of view or opinions stated do not, therefore, necessarily represent official Office of Education position or policy.

THE UNIVERSITY OF CHICAGO

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Thanks are due to Professors Marshall H. Stone and Saunders MacLane for their time and judgment generously given to examine the mathematical results on which this work was based.

As principal cooperating teacher, Mrs. Patricia Walker Hindman gave counsel and guidance to the proceedings, as well as identifying areas of investigation to which her class was ready to contribute. In fact the entire Department of Mathematics of the University High School is owed a debt of thanks for devoting a considerable number of departmental meetings to preliminary planning and screening of ideas. Its Chairman, Miss Lenore John, and its Consultant Member, Dr. Margaret S. Matchett, were especially helpful.

We are in debt to Professor Alice G. Hart of the Department of Mathematics at the University of Illinois Chicago Circle, and to Mr. Donald Lund, Mr. James Maxey, and Mr. Budd Bodine of the Department of Mathematics at Homewood-Flossmoor High School for their participation in the evaluation discussions which followed units of classroom experimentation.

Mr. Willard J. Congreve, Assistant Director of the Laboratory Schools and Principal of the University High School during this study, took valuable time from his duties to observe the classroom proceedings and to write his evaluations which are included herein.

Thanks are also due to the many staff members of the Laboratory School whose help contributed to the success of this project.

Finally, this project was possible only through the encouragement of Dean Roald F. Campbell of the Graduate School of Education of the University of Chicago, and his predecessor, Dean Francis S. Chase, and Francis V. Lloyd, Jr., Director of Pre-Collegiate Education, who provided support to this project from its inception, before there was any assurance of its funding.

INTRODUCTION

The past decade has seen curriculum improvement projects on an unprecedented scale. Their most noteworthy accomplishments have been, up to now, in altering the climate of receptiveness for change. The reform movement known as "New Math" has provided a framework and a rationale to support a more abstract outlook on the structure of arithmetic, and has introduced elements of geometry, algebra, and logic at very early levels.

But this is only a first step, and the many major curriculum improvement projects which have brought about this change of climate and outlook have helped to point out the need for a far more ambitious upgrading of the intellectual and factual content for the elementary mathematics curriculum. It is unfortunate that one concurrent trend has been the downgrading of common arithmetic for the purpose of providing time and effort for concepts required to support the more sophisticated abstract mode of development of mathematical skills and understanding.

By and large this has merely provided alternate means of attaining goals within the elementary curriculum which are not significantly greater than those attained in the curricula which have been subjected to improvement. Nevertheless, it is clear from some of the published deliberations of highly influential professional groups and individuals ([2], [5], [9], [10])[1] that improvement in content of a substantial character will be required to realize the anticipated goals of the upgraded secondary curriculum. We must be prepared for the fact that an improved curriculum which calls for calculus as a ninth or tenth grade course will also require that the elementary curriculum in arithmetic provide a foundation adequate to support such an enterprise.

It was for this purpose that this study was undertaken. The objective was to explore within the elementary mathematics curriculum how to attain a working competence in the arithmetic of the full real number system, and to explore means of removing some of the major obstacles, still existing within the improved versions developed in recent years, that stand in the way of this objective.

[1]Numbers in square brackets refer to the list of references, p. 15

The program whose feasibility was studied in this project had been developed in previous years by its proposers. Its essential components had been tried out at various points of the secondary and college levels, but its viability at the elementary level remained to be established.

The central principle of the program is that the most natural route to the arithmetic of the reals is through the infinite decimals. This approach lacks appeal to those who feel that abstraction is the essence of mathematics, but it has proven very effective to exploit the special concrete features of a specific representation of the reals such as the decimals. The material had been tried out in experimental courses in analysis for graduate students in science [5] [6], college calculus courses, and in a junior high school course, all at the University of Chicago. It was also presented in a television series, produced by the Midwest Program for Airborne Television Instruction, consisting of 64 half hour lessons designed as pre-calculus course for junior and senior high school students. There was a course manual prepared and published [7] which outlines the approach.

Arrangements for the conduct of this study were made with the cooperation of the Mathematics Department of the University of Chicago Laboratory Schools. It was of course necessary to proceed so that the interests of inquiry and students' needs did not conflict. Fortunately no such conflict arose; in fact, the adaptation of our material to the needs of the students resulted in considerable improvement of the material. There were three major areas in arithmetic that were treated at the suggestion of the teacher, Patricia Walker Hindman, of the combined 7th and 8th grade (Prefreshman) class that was selected for the study. These were the treatment of signed integers, fractions, and infinite decimals. Mrs. Hindman's description of the class and her comments on the significance of the trials is included later in this report.

Two other groups of pupils also participated in this study. One was a selection of fifth graders, available because of a scheduling situation. These pupils helped explore a means of developing the infinite decimal independently of fractions, using some commonplace mechanical devices as analog computers of sorts.

The third group consisted of some sixth graders attending the neighboring Kenwood Elementary School. They were members of Kento, a local church after-school organization, sponsoring educational enrichment activities for neighborhood children. This group verified a ready source of interest in infinite decimal arithmetic among the general school population.

METHOD

This project, being experimental in character and limited in time and resources, was able to develop only informal objective instruments of evaluation. But many expert observers were able to examine the various aspects of the procedures and evaluate the effectiveness of the program tested. The judgments expressed, often subjective, were a positive influence in the course of the study.

Prior to the project itself, Professors Saunders MacLane and Marshall H. Stone examined the theoretical foundation for the treatment of the reals as developed in this program and were able to assure us of its validity. Professor Stone included a statement regarding this program in his critique of the Cambridge Report ([10], p. 359). Also, prior to the project itself, during the year (1963-1964) when the proposers were conducting an experimental junior level course at the University of Chicago High School, a series of department meetings were devoted to reviewing the contents and results attained in that course. Also discussed were the implications of that experience for design of the curriculum, particularly matters that were recurrent concerns of the Department. Members of the Mathematics Department of the Middle and High Schools who participated in those discussions were: Miss Pamela Ames, Mr. Max Bell, Miss Betty A. Cacioppo, Mrs. Carol Dalrymple, Miss Margaret Eason, Miss Maryanne Garbarino, Mrs. Patricia Hindman, Miss Lenore John, Mrs. Gladys Junker, Mrs. Rachel Larson, Miss Elizabeth J. Lawrence, Dr. Margaret Matchett, Mr. Lynn Meedon, Mr. Paul Moulton, and Mr. Richard Muelder.

There was general agreement that three troublesome matters were creating mounting pressure. One was the delay in introducing signed numbers and the awkardness in extending the rules of arithmetic to them, particularly the rule of signs in multiplication. A second matter of concern was the low level of achievement in the arithmetic of fractions, and the prohibitive effort needed to develop from them a model of the reals along the lines of Dedekind cuts.

The second curriculum problem above led to the formulation of the main and third objective. This was to determine the feasibility of developing infinite decimal arithmetic directly, without building on the arithmetic of fractions. However expedience made it necessary to develop the decimals from fractions rather than directly from integer arithmetic. This was because the predominate

share of the classroom work was with an advanced section of the Pre-freshman class taught by Mrs. Hindman with which the project previously had developed some extra facilities with fractions.

Mrs. Hindman, who participated in the discussions along with other members of the University High School Mathematics Department, felt that her class would be a particularly fitting group with which to test the program that the proposers had outlined during the meetings. Accordingly, this class was the principal testing group.

There were two other groups of pupils who participated in this study. One was a group of fifth graders new to the Laboratory Schools who, because they lacked adequate preparation in the foreign languages studied by the rest of their classes, were at loose ends during the period set aside in their schedule for language study. These were turned over to Mr. Klein who worked with them in developing the introduction to infinite decimal arithmetic independently of fraction facts and skills.

The second of these groups was made up of pupils in an after-school program for public school sixth grade pupils from Kenwood, a neighborhood adjacent to the University of Chicago. This group was organized by Miss Sharon Friedman of the Laboratory Schools. The children volunteered for this program of enrichment activities and Mr. Klein presented some of the work in infinite decimals with them.

The procedure used with Mrs. Hindman's class was to discuss with her the background of the class and to assess the pupil skills that could be utilized as a base. We used rough screening tests and class exercises to determine these skills.

Each unit of work was planned jointly by Mrs. Hindman and Mr. Klein. The classroom procedure was based on worksheets prepared before each class.

The classroom procedure was informal and pupils were given time to make comments on their reaction to the material. Mrs. Hindman's evaluation of this part of the program is included in the Discussion Section below. As the work with the pupils proceeded Mrs. Hindman and Mr. Klein discussed the classroom progress informally with the members of the Mathematics Department and reviewed the project more formally at the regular Department meetings.

During the quarter (January - March 1965) in which this experimental program was tested there were a number of visitors

to the class both from within the University and from outside. Both Mr. Francis V. Lloyd, Jr., Director of Pre-Collegiate Education, and Mr. Willard Congreve, Assistant Director of the Laboratory Schools and Principal of the High School, visited the class in session. Mr. Congreve's comments and observations are given below in the Discussion section. Other interested persons from the Graduate School of Education also visited the class.

Visitors from outside the University included Dr. Lauren G. Woodby of the U. S. Office of Education, Professor Alice G. Hart of the University of Illinois at Chicago Circle, and Professor Taffee Tanimoto, Chairman of the Department of Mathematics, University of Massachusetts, Boston.

The progress of the project was discussed in several meetings attended by members of the University High School Mathematics Department, and other interested teachers at the Laboratory Schools, by Mr. Budd Bodine, Mr. Don Lund, and Mr. James Maxey of Homewood-Flossmoor High School, and by Dr. Lauren G. Woodby, and Professor Alice G. Hart. During these meetings valuable suggestions and observations were made which benefited the program and aided in its evaluation.

The procedure followed with the fifth grade group was less formal: Mr. Klein worked alone. The fifth grade teachers, Mrs. Carol Dalrymple, Miss Betty Ann Cacioppo and Miss Elizabeth Jean Lawrence, were able to visit the sessions frequently and supplied background information regarding the individual pupils participating.

The evaluation of this phase of the project is very subjective, but these teachers were able to concur in the evaluation of the program as discussed with them by Mr. Klein.

The work with the Kenwood after-school group was even more informal and the testing was very exploratory. These children were sixth graders visiting the Laboratory Schools. These visits were once a week for several months. They were a part of a voluntary enrichment program called Kento, organized by Miss Sharon Friedman of the Laboratory Schools, in cooperation with a neighborhood church community group. The pupils also visited the Computation Laboratory of the Graduate School of Education. They used some of the mechanical devices and problem material also used with the other groups. Mrs. Fay Abrams, a fourth grade teacher at the Laboratory Schools, observed most of these meetings. The evaluation of this phase of the project should be considered subjective except for the worksheets that were retained for later examination.

RESULTS

The results of this project are based on the level of performance and development of mathematical sophistication on the part of the participating pupils. Time and personnel resources were not available to design test instruments to measure this type of achievement statistically. But there is no doubt in the mind of the investigator that judgments of the experienced teachers participating in this study represent a more reliable basis for evaluating the significance of the experiments.

Mrs. Hindman prepared written tests based on the work in her class which she used for grading purposes and assessing the effects of the classroom work on her pupils. Her report is given in the next section.

Some impression of the character of the work done in Mrs. Hindman's class can perhaps be best conveyed by a topical description of the material tested:

Signed Integers. Three independent interpretations of signed integers were developed: location indicators on a number line, transformations on the number line -- directed translations, and as arithmetical additive inverses. The equivalence of these notions was developed through exercises showing the consistency and dependability of results obtained by interchanging the notion on which the computations were made. A side product of this work was the rule of signs in multiplication. This rule was also verified by reference to the change in area of a rectangle made up of unit squares, when the sides are reduced.

The classroom work showed that the contrast of several points of view not only gave each student a choice of approach, but showed that there is little basis for not introducing negative integers much earlier in the curriculum.

Fractions. This material represents a significant departure from the usual treatment of this subject, which traditionally is based on multiplication facts. The development was based on new methods of obtaining fractions equal to given ones - the new ones fulfilling a variety of specifications. The principal novelty centered on means of obtaining these equivalents by changing numerators and denominators through addition rather than multiplication.

Decimal Arithmetic. The above ideas became powerful instruments in establishing a smooth link to finite and infinite decimals, and

led to quite interesting arithmetical procedures. Mathematically significant constructions such as infinite continued (numerator) fractions were also developed: the core ideas also provided a handily available equivalence of the fractions and repeating decimals which is quite complete. As an illustration of this we cite the result obtained that every rational number has a fraction equivalent of the form N/99..90..0 (where N is an integer). For the latter, we had routines that efficiently produced the repeating decimal equivalent.

The infinite decimal arithmetic was of course our main interest and the preparatory work in fractions led very naturally to this material. One highly significant result of this phase of the project is that a complete infinite decimal arithmetic can be built and realized within the elementary curriculum. In the Pre-freshman class we developed this from fraction facts of a more comprehensive character than is usually provided. But another phase of the study was to show that the introduction of the infinite decimal arithmetic was also feasible by an approach independent of fraction facts. This means that much of the work on infinite decimals can be initiated by the fifth grade if not earlier.

Topics developed in the Pre-freshman group ranged significantly further than suggested by the outlines included in the Cambridge Report 15. These include the European Multiplication Algorithm for finite decimals (where the order of multiplication by the digits of the multiplier are reversed). This style of multiplication generalizes neatly to the case of infinite decimal multiplicands and multipliers, while the conventional procedure does not. It also is more inducive to correction methods (making allowances for changes in individual digits of the multiplier and multiplicand) and is closely akin to techniques of rapid mental computation.

The correction method can be used for a trial and error evolution of square roots: in fact we followed this procedure to formulate and justify the usual square root algorithm. Applying this technique to cube roots, we developed an improved cube root algorithm. The pupils were thus treated to a participation in genuine mathematical research.

A very pronounced interest was achieved by the Pre-Freshman pupils in every discussion or occasion where the notion of infinity played a part. A particularly touchy issue concerned the equation, $.\overline{9} = 1$. The most acceptable grounds of support proved to be an assessment of the consequences to the familiar facts of

arithmetic that followed from its denial. Formal arguments, appeals to higher authority such as limit theory and approximation arguments were far less convincing to the pupils. It was quite clear that the notions held by pupils (even the more highly achieving ones) regarding the order structure of the real line are extremely confused, and that the concreteness of the decimal representation was an incisive tool of clarification aside from the role of decimals in arithmetic.

The arithmetical competence developed was very far-reaching. The pupils were able to demonstrate as a class project their ability to add two infinite non-repeating decimals, never having encountered a problem of this nature before. The problem was formulated with the help of a consecutive number stamper, and the pupils were quite able to express the answer in a mathematically complete form. This class experiment was observed by Mr. Lloyd, Director of Pre-Collegiate Education.

Pupils were able to add infinite arrays of infinite decimals. These arrays were often more complicated than infinite series. Through its mechanical aspects of the schema, the multiplication of infinite decimals was shown to be well determined. Other infinite expression complexes such as

$$\sqrt{1 + \sqrt{2 + \sqrt{3 + \sqrt{\ldots}}}}$$

were discussed. Continued fraction methods for evolving solutions to exponential equations such as $2^x = 3$ were also treated without surpassing the interest or understanding of the class.

The completeness property of the real number system is a central tool in this arithmetic program. That the infinite decimal representation of the real numbers makes possible a mechanical demonstration of this fact is a critical step in establishing the real numbers by constructive rather than postulational methods. A key result of this study was the full feasibility of obtaining this completeness in a mathematically rigorous way and within the understanding of the elementary school student.

The work of the fifth grade group was to show the feasibility of developing the infinite decimal notation and usage directly from integer arithmetic and in a form mathematically independent of fractions. Mechanical devices proved to be very helpful in this regard, both in learning systems of notation and in guiding their operational usage. Devices used included U. S. and English currency, number stampers, number counters and a desk calculator.

Children readily learned to convert between the two money systems and worked with conventional English notation as well as one invented for U. S. coin combinations (e.g., 11203 denoting 1 half dollar, 1 quarter, 2 dimes, etc.). Use of the desk calculator, an early version Monroe, with a hand operated carriage shift, was particularly useful for suggesting infinite repeating decimals as an end product in long division.

A result important to the study was the increased capability acquired by the pupils to design and adapt various systems of notation for special purposes. Miss Cacioppo's observations of this group is given in the Discussion Section below.

Attendance of the after-school group of sixth graders (Kento) was not consistent so that programs of the various sessions could not be cumulative or formal. A positive result of the work with this group is that successful sixth grade pupils are capable of work in adding and multiplying repeating decimals, without prior stimulation of their interest, or special preparation of skills. The mechanical gadgetry very much engaged their attention, particularly a gasoline computer mechanism newly acquired.

DISCUSSION

The results reported above do not require further interpretation. They are quite subjective in character and made by an investigator who cannot claim to be completely detached.

An important factor in the feasibility of any program of this nature is the extent of student interest in the subject matter and teacher assessment of the value of the program. These are aspects difficult to measure quantitatively, so the judgment of experienced and knowledgeable observers is important. The following commentaries are included, not as testimonials, but as evidence of the value of the program in the eyes of highly skilled educators, very much concerned for the interests of their pupils.

The teachers, Miss Cacioppo and Mrs. Hindman, whose comments are quoted below were involved with the project over a prolonged period and were thoroughly familiar with the backgrounds of the participating pupils. Mr. Congreve is an authority on curriculum evaluation whose opinion is highly valued.

Mrs. Hindman:

During the 1964-65 school year Mr. Klein and Mr. W. H. L. Meyer taught a group of juniors at The Laboratory Schools. They reported periodically to the members of the Mathematics Department about what they were doing in this class. It seemed apparent to both Mr. Klein and Mr. Meyer that there was much similar material which could be done with younger students - perhaps students as young as the 12 and 13 year olds I taught. During the next school year at my invitation Mr. Klein and Mr. Meyer observed in my classroom, made suggestions to me about possible topics and approaches, and Mr. Klein actually spent some weeks teaching about infinite decimals. I was delighted with the results and asked Mr. Klein to consider spending one academic quarter of the following school year in my class.

In my class there were 23 twelve and thirteen year olds - chosen because they were either good mathematics students and/ or particularly interested in mathematics. Mr. Klein and I worked together with these students during the winter quarter of the academic year 1966-67. He took the initiative for determining the course of events in the class. We worked together writing and mimeographing materials. I supervised extra sessions of conference hours. Equal fractions and infinite decimals were the two basic units. To make sure that each student understood the material, we gave periodic short quizzes.

The results of the quarter's work were encouraging. The students voiced a positive liking for the material, and from my point of view as a teacher, this interest became obvious to me not just by the students' comments but by their willingness to work diligently both at school and at home. In part this diligence and basic interest came from the suitability of the material for the students. Junior high school students are fascinated by patterns and the infinite, and these two ideas are the backbone of the unit on the infinite decimals. Also, junior high school students still need practice with fractions and decimals, but they are quite bored by the usual problems of drill. This material, by approaching fractions and decimals from quite a different point of view, offered the much needed practice.

Aside from providing a positive climate of interest and motivation, the material extended far beyond the usual curriculum offerings in arithmetic. Pupils showed an interest and

aptitude for this part of the program. They were able to master and devise direct and indirect arguments; work with very advanced problems in arithmetic such as multiplication of infinite decimals, irrationality of $\sqrt{2}$, and exponential equations such as $2^x = 3$. The treatment of these problems involved the completeness property of the infinite decimal system which was a central issue for testing. The students were quite able to understand the use of this property and appreciate its significance.

Mr. Congreve, Principal of the University High School visited Mrs. Hindman's Advanced Mathematics section on January 7, 1966 to observe the experimental class at work. The following is a summary of his remarks to Mr. Klein describing his reaction to the class.

1. Impressed by industry of pupils: no evidence of any not involved.
2. Impressed by excitement of pupils.
3. Fundamental concepts required were well understood; tools and background accessible so that pupils could resolve issues and questions they raised themselves.
4. There seemed to be none of the usual fear of fractions and decimals; pupils exhibited no hesitancy in using fractions or decimals in numerators and denominators, even in mixed cases.
5. Material has some aspect of being programmed effectively: pupils working independently and at different paces; no formal instruction, just individual conferences among pupils and pupils with teachers.
6. Class is action oriented, pupils obviously learning from their own work; not as result of lecture by teacher.
7. Looked for pupils in trouble but found none.

Mr. Congreve would like to visit the class again to see if the achievements of the pupils are incidental or are part of larger goals. That is, pupils are responding well to new and challenging material, but so what and where is it leading to? One visit not enough to gauge this by independent observation.

Mr. Congreve generally was very favorably impressed. The atmosphere was pleasant and no stress was observed. Material interesting and in a format that is effective.

Miss Cacioppo:

This group of children could not participate in the foreign language program because they needed extra work in other subjects, or because they were new to the school and

could not begin the foreign language program at their grade level. The majority of the children were in the latter category. The fifth grade teachers felt that the project developed gave a new direction to mathematics for most of the children in the group.

Two children who had a school history of being slow to grasp mathematical concepts found success with Mr. Klein, and for a while really did not think that they were doing mathematics, since it had previously proved so difficult for them. This was particularly true for one boy when working with English money. For another child, it helped her to develop a sense of confidence in mathematics that lasted through the year. Another boy, always quiet in mathematics, was willing to get up in front of each fifth grade class to explain how one of the machines worked. Speech making was difficult for him, but he really liked the machine and wanted it run properly.

There was a good deal of carryover for the other children in many different ways. Through conversation about the activities, other children wanted to become part of this group too. It may have been due to their dislike of French or German, but I really think there was quite a bit of envy of those children having an extra mathematics period.

CONCLUSIONS

The principal conclusion of this study is that it is feasible to attain within the limits of the elementary curriculum a working mastery of the real number system by appropriate use of its infinite decimal representation as a vehicle.

The study also showed that this arithmetic can be built on or developed independently of fraction facts.

Time needed for inclusion of this material in the curriculum is available if improvements in the treatment of the arithmetic of signed numbers and fractions is exploited. Another resource is to be found in the use of mechanical devices which are readily available and which can serve as analog computers of a sort. These are especially effective for a program that develops the infinite decimals independently of the fractions.

Regardless of which system is developed first, the infinite decimals and the fractions complement the arithmetic of each other in valuable ways. Tools of one system determine results in the other in a manner that can be pursued by elementary school pupils.

The implications of these conclusions for curriculum design at the elementary and secondary levels are far reaching. For one, it shows that the goals of the Cambridge Report are not unattainable, if experience and skill in arithmetic are made available to advance the mathematical maturity and sophistication of the elementary school pupil. These higher levels of attainment can be realized in less time than is now devoted to arithmetic. Time and opportunity to introduce geometrical and algebraic notions and skills can also be provided by a more comprehensive treatment of arithmetic.

These implications are of course in direct contrast to many of the tendencies in the major curriculum improvement programs of recent years. The experience of this investigator is that the downgrading of arithmetic that is typical of these programs increases to the danger point the dependence of the learner on prior decisions regarding the definitions, laws, and axioms that are necessary to the abstractionist point of view.

SUMMARY

Reform in the elementary mathematics curriculum in recent years has encouraged a new receptiveness for change, both in organization and in content in many parts of the world ([1]). Much new material in algebra, geometry, and logic has been introduced along with the concepts and philosophy of the "New Math". But the trends have received criticism as well as encouragement from high quarters ([3], [4]); much of the concern is about the role and treatment of arithmetic. It is widely recognized that an effective model of the real number system is needed as a basis for a significantly improved secondary mathematics curriculum ([2], [5], [9], [10]), not only to clarify the theoretical foundations for algebra and geometry, but to make possible a suitable high school course in calculus.

There are various ways to establish such a model. The traditional method is through the rationals and Dedekind cuts. This not only is an exhausting program to complete, but is decidedly unappealing and unsuitable to school mathematics. This project

was designed to test the viability of a more concrete representation of the reals in the form of infinite decimals, within the limitations of the elementary mathematics curriculum. Part of this feasibility test was to investigate means of removing some obstacles in the usual presentation of arithmetic that take up much valuable time. Two of those dealt with were the arithmetic of signed integers and the arithmetic of fractions.

The theoretical basis for the program was examined earlier by Professors Saunders MacLane and Marshall H. Stone ([10]) of the Department of Mathematics at the University of Chicago. The program itself was tried out earlier at various higher levels in the curriculum, ranging downward from post-graduate, undergraduate, and high school levels at the University of Chicago. With the cooperation of the University High School Mathematics Department and support of the Graduate School of Education it was possible to verify that it is feasible and within the resources of the elementary mathematics curriculum:

1) to extend arithmetical competence to a working mastery of the complete decimal model of the real number system;
2) to accomplish this goal without exhausting the interest and motivation of the pupils;
3) to improve the treatment of topics such as the arithmetic of the signed integers and fractions which presently are troublesome and time consuming; and
4) to develop the infinite decimals directly from mechanical instruments and notation, or from fraction facts suitably adapted and extended.

The evaluations stated above are subjective in character and are presented as the conclusions of the chief investigator of this project, George Klein, by no means a detached observer. However, the progress of this study was at all times under observation and counsel of experienced and skilled teachers, administrators, and authorities in this subject area. They are not to be held responsible for the conclusions as stated but their observations give some assurance of their validity.

A significant feature of the program tested is its decidedly non-postulational character. As a matter of principal, axioms and other paraphernalia of the abstractionist viewpoint are avoided as being irrelevant to proper educational goals of the elementary curriculum. The notations and algorithms are as concrete as possible, but nevertheless results of wide generality and sophistication are important components of the program.

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